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$$\int_0^{2\pi}\frac{\cos^2\zeta d\zeta}{2\pi\omega}=\frac{1-\nu}{2\mu}; \qquad \int_0^{2\pi}\frac{\sin^2\zeta d\zeta}{2\pi\omega}=\frac{1+\nu}{2\mu};$$

$$\int_0^{2\pi}\frac{\cos^2\zeta d\zeta}{2\pi\omega^3}=\frac{1+\nu}{2m^2\mu}; \qquad \int_0^{2\pi}\frac{\sin^2\zeta d\zeta}{2\pi\omega^3}=\frac{1-\nu}{2n^2\mu};$$

$$\int_0^{2\pi}\frac{d\zeta}{2\pi\omega^3}=\frac{1+\nu}{2m^2\mu}+\frac{1-\nu}{2n^2\mu}=\frac{m^2+n^2-(m^2-n^2)\nu}{2m^2n^2\nu};$$

and the integrals from 0 to $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ and 2π are always proportional to them.

The indefinite integrals

$$\int\frac{d\zeta}{\omega}=\frac{\Theta}{\mu};$$

$$\int\frac{(\cos^2\zeta-\sin^2\zeta)d\zeta}{\omega}=-\frac{\nu\Theta}{\mu}+\frac{\lambda_1\cos\zeta\sin\zeta_1+\lambda_2\cos\zeta\sin\zeta_2+\&c.}{\lambda^2}=-\frac{\nu\Theta}{\mu}+\Lambda;$$

$$\int\frac{\cos^2\zeta d\zeta}{\omega}=\frac{(1-\nu)\Theta}{2\mu}+\frac{\Lambda}{2}; \quad \int\frac{\sin^2\zeta d\zeta}{\omega}=\frac{(1+\nu)\Theta}{\mu}-\frac{\Lambda}{2};$$

$$\int\frac{\cos^2\zeta d\zeta}{\omega^3}=\frac{(1+\nu)\Theta+\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2m^2\mu}; \quad \int\frac{\sin^2\zeta d\zeta}{\omega^3}=\frac{(1-\nu)\Theta-\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2n^2\mu};$$

$$\int\frac{d\zeta}{\omega^3}=\frac{\left(m^2+n^2-(m^2-n^2)\nu\right)\Theta-(m^2-n^2)\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2m^2n^2\nu};$$

where ν, Λ, &c., have the values given above.

THE SUMMATION OF SERIES WHOSE COEFFICIENTS FORM AN ARITHMETICAL PROGRESSION OF ANY ORDER.

BY PROF. J. SCHEFFER, MERCERSBERG, PA.

LET the series be represented by $ax+bx^2+cx^3+dx^4+\ldots$ whose sum we denote by S. Putting $x=y\div(1+y)$, we have

$$S=a\left(\frac{y}{1+y}\right)+b\left(\frac{y}{1+y}\right)^2+c\left(\frac{y}{1+y}\right)^3+d\left(\frac{y}{1+y}\right)^4+\ldots\ldots$$

Developing each of these fractions by the Binomial Theorem, we have

$$S=ay-(a-b)y^2+(a-2b+c)y^3-(a-3b+3c-d)y^4+(a-4b+6c-4d+e)y^5-\ldots$$

We see that the coefficients of this series are the successive finite differences, which we shall denote by Δa, Δ^2a, Δ^3a, ..., consequently

$$S=ay+\Delta a.y^2+\Delta^2a.y^3+\Delta^3a.y^4+\ldots \text{ or}$$

$$S=a\left(\frac{x}{1-x}\right)+\Delta a\left(\frac{x}{1-x}\right)^2+\Delta^2a\left(\frac{x}{1-x}\right)^3+\Delta^3a\left(\frac{x}{1-x}\right)^4\ldots$$

As the coefficients form an arithmetical progression, one of these differences and all the following will become $= 0$.

This method is very useful in finding the sum of a great many series.

For instance; let $S = x+3x^2+5x^3+7x^4+9x^5+\ldots$ to infinity. Here $a = 1$, $\Delta a = 2$, $\Delta^2 a = \Delta^3 a = \ldots = 0$. Hence

$$S = \frac{x}{1-x}+2\left(\frac{x}{1-x}\right)^2 = \frac{x(1+x)}{(1-x)^2}.$$

In the series

$$S = x+4x^2+9x^3+16x^4+25x^5+\ldots \text{ ad inf.},$$

we have $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, $\Delta^3 a = \Delta^4 a = \ldots = 0$. Therefore

$$S = \frac{x}{1+x}+3\left(\frac{x}{-x}\right)^2+2\left(\frac{x}{1-x}\right)^3 = \frac{x(1+x)}{(1-x)^3}.$$

According to this method we can also find the sum of a given number of terms. Putting

$$S' = ax+bx^2+cx^3+dx^4+\ldots+lx^n+px^{n+1}+qx^{n+2}+\ldots.$$

and $S = ax+bx^2+cx^3+dx^4+\ldots+lx^n$, we have

$$S = S'-x^n(px+qx^2+rx^3+\ldots)$$

$$= S'-x^n\left[p\left(\frac{x}{1-x}\right)+\Delta p\left(\frac{x}{1-x}\right)^2+\Delta^2 p\left(\frac{x}{1-x}\right)^3+\ldots\right],$$

but $S' = a\left(\frac{x}{1-x}\right)+\Delta a\left(\frac{x}{1-x}\right)^2+\Delta^2 a\left(\frac{x}{1-x}\right)^3+\ldots$; therefore

$$S = (a-x(p)\frac{x}{1-x}+(\Delta a-x^n\Delta p)\left(\frac{x}{1-x}\right)^2+(\Delta^2 a-x^n\Delta^2 p)\left(\frac{x}{1-x}\right)^3 + \ldots.$$

Ex. 1. In the common geometrical progression, we have

$$ar+ar^2+ar^3+\ldots.+ar^n;$$

where $a = a$, $\Delta a = 0$; $p = a$, $\Delta p = 0$; hence

$$S = (a-ar^n)\frac{r}{1-r}.$$

Ex. 2. $S = x+2x^2+3x^3+4x^4+\ldots.+nx^n$. $a = 1$, $\Delta a = 1$, $p = n+1$, $\Delta p = 1$; hence

$$S = [1-(n+1)x^n]\frac{x}{1-x}+(1-x^n)\left(\frac{x}{1-x}\right)^2 = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}.$$

Ex. 3. $S = x+4x^2+9x^3+16x^4+\ldots+n^2x^n$. $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, $p = (n+1)^2$, $\Delta p = 2n+3$, $\Delta^2 p = 2$; therefore

$$S = [1-(n+1)^2x^n]\frac{x}{1-x}+[3-(2n+3)x^n]\left(\frac{x}{1-x}\right)^2+(2-2x^n)\left(\frac{x}{1-x}\right)^3$$

$$= \frac{-x^{n+1}[n(x-1)-1]^2+x(x+1)-x^{n+2}}{(1-x)^3}.$$

Putting $-x$ in place of x, we have

$$S = ax-bx^2+cx^3-dx^4+\ldots.$$

$$= a\left[\frac{x}{1+x}\right]-\Delta a\left[\frac{x}{1+x}\right]^2+\Delta^2 a\left[\frac{x}{1+x}\right]^3-\ldots$$

Putting $x = 1$, we have

$$a-b+c-d+e-f+\ldots \text{ ad inf.} = \tfrac{1}{2}a-\tfrac{1}{4}\Delta a+\tfrac{1}{8}\Delta^2 a-\tfrac{1}{16}\Delta^3 a+\ldots.$$

Thus we find, $1-1+1-1+1-1+1-1+\ldots$ ad inf. $= \frac{1}{2}$,

$1-2+3-4+5-6+7-8+\ldots$ ad inf. $= \frac{1}{4}$,

$1-3+5-7+9-11+13-\ldots$ ad inf. $= 0$,

$1-4+9-16+25-36+49-\ldots$ ad inf. $= 0$.

The formula $ax-bx^2+cx^3-dx^4+\ldots$

$$= a\left[\frac{x}{1+x}\right] \quad \Delta a\left[\frac{x}{1+x}\right]^2+\Delta^2 a\left[\frac{x}{1+x}\right]^3-\ldots$$

is also very useful, sometimes, to change a series into one that converges more rapidly. For instance,

$$\log(1+x) = x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots.$$

$$= \frac{x}{1+x}\left\{1+\frac{1}{2}\left[\frac{x}{1+x}\right]+\frac{1}{3}\left[\frac{x}{1+x}\right]^2+\frac{1}{4}\left[\frac{x}{1+x}\right]^3+\ldots\right\}.$$

$$\tan^{-1}x = x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\ldots = \frac{1}{x}(x^2-\tfrac{1}{3}x^4+\tfrac{1}{5}x^6-\tfrac{1}{7}x^8+\ldots)$$

$$= \frac{x}{1+x^2}\left\{1+\frac{2}{3}\left[\frac{x^2}{1+x^2}\right]+\frac{2.4}{3.5}\left[\frac{x^2}{1+x^2}\right]^2+\frac{2.4.6}{3.5.7}\,\&\text{c.}\right\}.$$

The above method of the summation of series is by *Euler*. He published it in his Inst. Calc. Diff.

Query by E. Hancock, Elgin, Ill. — If a wheel of 60 teeth work in a pinion of 10 leaves, the distance between their centres being .455 in., the diameters of the pitch circles will be .78 in., $= a$, and .13 in., $= b$, respectively. Allowing one-half for space between the teeth, each tooth will occupy a central angle of 3°, $= c$. Suppose the teeth to terminate at the intersection of two similar epicycloidal curves the diameter of whose generating circle is .065, $= \frac{1}{2}b$; what must be the *addendum* to the wheel outside of the pitch circle, i. e., what is the length of the teeth, outside of the pitch circle?